PROBLEM DESCRIPTION AND QUESTIONS

You are a young MEMS designer and your supervisor asks you to redesign a high performance accelerometer and to give some forecast on the mechanical performance of the sensor. The fixed geometrical parameters are reported in Table 1. The accelerometer must ensure a full scale range $FSR = 16\hat{g}$. The parallel plates are polarized at $V_{DD} = 3V$.

1. Given that the maximum acceptable linearity error, $\epsilon_{lin}$, is equal to 1%, calculate the maximum capacitance variation.

2. Calculate the mechanical sensitivity $[fF/\hat{g}]$ and the resonance frequency during the operation of the accelerometer.

3. Evaluate the contribution of the parallel plates in terms of electrostatic stiffness and choose the geometry of the springs.

4. Calculate the needed quality factor, $Q$, to guarantee a $NEAD = 25 \frac{\mu g}{\sqrt{Hz}}$.

INTRODUCTION

A 'MEMS accelerometer' is a microelectromechanical system used to reveal linear accelerations. MEMS devices are modeled with a spring-mass-damper system, in order to describe
their static and dynamic behavior. The equation of motion of such a system can be written as follows:

\[ m\ddot{x} + b\dot{x} + kx = F_{\text{ext}} \]

Being \( X(s) \) the Laplace transformation of the relative position and \( F_{\text{ext}}(s) \) the Laplace transformation of the external force, one can find the transfer function \( T_{XF}(s) \) as

\[ T_{XF}(s) = \frac{X(s)}{F_{\text{ext}}(s)} = \frac{1/m}{s^2 + \frac{b}{m}s + \frac{k}{m}} \]

Writing this equation in terms of \( j\omega \) and considering that \( \omega_0 = \sqrt{k/m} \) and \( Q = \omega_0 m/b \), we obtain

\[ T_{XF}(j\omega) = \frac{X(j\omega)}{F_{\text{ext}}(j\omega)} = \frac{1/m}{\omega_0^2 - \omega^2 + j\frac{\omega_0\omega}{Q}} \]

The relation between displacement and force can be simplified in three different ways, depending on the value of \( \omega \):

- \( \omega \ll \omega_0 \)

\[ |T_{XF}(j\omega)|_{\omega\ll\omega_0} = \frac{1/m}{\omega_0^2} = \frac{1}{k} \]

this means that when a quasi-stationary force is applied to the seismic mass the displacement is governed just by the stiffness constant,

- \( \omega = \omega_0 \)

\[ |T_{XF}(j\omega)|_{\omega=\omega_0} = \frac{1/m}{\left(\frac{\omega_0^2}{Q}\right)^2} = \frac{Q/m}{\omega_0^2} = \frac{Q}{k} \]

this means that applying a force at the resonance frequency of the device, the displacement increases by a factor \( Q \) with respect to the quasi-stationary case,
\( \omega \gg \omega_0 \)

\[ |T_X(j\omega)|_{\omega \gg \omega_0} = \frac{1}{\omega^2} \]

this means that over the resonance frequency the displacement is proportional to the inverse of the frequency modulation of the applied force. Accelerometer typically works in the first frequency range, with input acceleration frequencies smaller than \( \omega_0 \).

**QUESTION 1**

The parallel plate configuration is common in accelerometers because of the higher sensitivity with respect to a comb finger configuration at equal area occupation. This configuration suffers however from geometrical non-linearity: this phenomenon is described by the percentage linearity error \( \epsilon_{\text{lin}} \). From the specifications, the acceptable value of this parameter turns out to be 1% and is defined as:

\[ \epsilon_{\text{lin}} = \frac{\Delta C_{\text{real,FSR}} - \Delta C_{\text{lin,FSR}}}{\Delta C_{\text{real,FSR}}} \cdot 100 \]

where \( \Delta C_{\text{real,FSR}} \) is the real capacitance variation at the full scale range between the capacitance \( C_1 \) and \( C_2 \), computable as:

\[
\Delta C_{\text{real,FSR}} = C_2 - C_1 = \varepsilon_0 N_{pp}h \frac{l_{pp}}{g-x} - \varepsilon_0 N_{pp}h \frac{l_{pp}}{g+x} = \varepsilon_0 N_{pp}h l_{pp} \left( \frac{1}{g-x} - \frac{1}{g+x} \right) = \varepsilon_0 N_{pp}h \frac{l_{pp}}{g} \left( \frac{1}{1 - \frac{x}{g}} - \frac{1}{1 + \frac{x}{g}} \right) = C_0 \frac{2 \frac{x}{g}}{1 - \left( \frac{x}{g} \right)^2}
\]

Were \( C_0 \) is the “rest” capacitance, i.e. with no displacement applied to the MEMS accelerometer. Note the multiplication factor \( N_{pp} \), arising from the fact that there are 10 differential parallel plate cells.

On the other hand, using the small displacement approximation, \( x \ll g \), one can obtain the linearized expression expression for \( \Delta C_{\text{lin}} \):

\[ \Delta C_{\text{lin}} = 2C_0 \frac{x}{g} \]

Imposing \( \epsilon_{\text{lin}} = 1\% \), one can fix the maximum acceptable displacement, reached at the full scale, \( x_{\text{FSR}} \):

\[
\epsilon_{\text{lin}} = \frac{C_0 \left| \frac{2 \frac{x}{g}}{1 - \left( \frac{x}{g} \right)^2} \right|_{\text{FSR}} - 2C_0 \frac{x}{g}}{C_0 \left| \frac{2 \frac{x}{g}}{1 - \left( \frac{x}{g} \right)^2} \right|_{\text{FSR}}} \cdot 100 \rightarrow \epsilon_{\text{lin}} C_0 \left| \frac{2 \frac{x}{g}}{1 - \left( \frac{x}{g} \right)^2} \right|_{\text{FSR}} = 100 \left( C_0 \left| \frac{2 \frac{x}{g}}{1 - \left( \frac{x}{g} \right)^2} \right|_{\text{FSR}} - C_0 \frac{x}{g} \right)
\]
\[
\begin{align*}
(100 - \epsilon_{lin}) \left| \frac{1}{1 - \left( \frac{x}{g} \right)^2} \right|_{FSR} &= 100 \\
\Rightarrow 100 - \epsilon_{lin} &= 1 - \left( \frac{x_{FSR}}{g} \right)^2 \\
\Rightarrow \left( \frac{x_{FSR}}{g} \right)^2 &= \epsilon_{lin} \\
\Rightarrow x_{FSR} &= g \sqrt{\frac{\epsilon_{lin}}{100}}.
\end{align*}
\]

Hence, the maximum displacement is governed by the linearity of parallel plates: in this case \( x_{FSR} = 200nm \) and the corresponding value of capacitance variation is \( \Delta C_{real,FSR} = 64.36 \text{ fF} \).

**QUESTION 2**

You have seen during the classes that the mechanical sensitivity of a parallel plates MEMS accelerometer can be expressed as:

\[
S_{mech} = \frac{1}{\omega_0^2} \cdot \frac{2 \cdot C_0}{g}
\]

Anyway, in our particular situation, we don’t have information about the accelerometer resonant frequency. There is an alternative (and much simpler) way to find the sensitivity when we have information about full scale range and linearity error. In fact, the mechanical sensitivity is defined as the capacitance variation per gravity constant unit. We already know the capacitance variation for an external acceleration \( a_{FRS} = 16\hat{g} \), so the sensitivity is given by the following ratio:

\[
S_{mech} = \frac{\Delta C_{real,FSR}}{FSR} = \frac{4.1 \text{ fF/}\hat{g}}{0.4 \text{ fF/}(m/s^2)}
\]

In order to evaluate the working resonance frequency, one can write the displacement in terms of applied force on the seismic mass through the transfer function \( |T| \), remembering that the accelerometer works at \( \omega \ll \omega_0 \).

\[
X = F \cdot |T(\omega < \omega_0)| \rightarrow x = \frac{F}{k_0} \rightarrow x = \frac{m \cdot a}{k} \rightarrow x = \frac{a}{\omega_0^2} \rightarrow \omega_0 = \sqrt{\frac{a_{FRS}}{x_{FSR}}}
\]

The obtained resonance frequency value is referred to the accelerometer in operation. The reader is invited to note that the only data used so far are the linearity error, the gap and the value of the full scale range. The resonance frequency results \( f_0 = 4456 \text{ Hz} \), since \( f_0 = \omega_0/2\pi \).

**QUESTION 3**

The considered sensor presents a capacitive readout based on parallel plates configuration. In order to evaluate the effect of electrostatic softening introduced by the electrostatic force of the parallel plates, one can rewrite the characteristic equation of the spring-mass-damper system:
\[ m\ddot{x} + b\dot{x} + kx = F_{ext} \quad \rightarrow \quad m\ddot{x} + b\dot{x} + kx = ma_{ext} + F_{elec} \]

Then, one can calculate the expression for the electrostatic force, \( F_{elec} \), in the case of differential parallel plates configuration.

\[
F_{elec} = F_{elec,2} + F_{elec,1} = \frac{1}{2}\frac{\partial C_2}{\partial x} V_{DD}^2 + \frac{1}{2}\frac{\partial C_1}{\partial x} V_{DD}^2 = \frac{V_{DD}^2}{2} \left( \frac{\partial C_2}{\partial x} + \frac{\partial C_1}{\partial x} \right) = \frac{V_{DD}^2}{2} \left[ \frac{1}{x^2} + \frac{1}{(x+g)^2} \right] = \frac{V_{DD}^2}{2} \frac{\delta}{\partial x} \left( \frac{1}{x^2} - \frac{1}{(x+g)^2} \right)
\]

In the case of small displacement approximation, \( x \ll g \), the electrostatic force, \( F_{elec} \), can be written in the simplified formula

\[
F_{elec} = \frac{V_{DD}^2}{2} \frac{C_0}{g} \left[ \frac{1}{1 - \frac{2x}{g} + \left(\frac{x}{g}\right)^2} - \frac{1}{1 + \frac{2x}{g} + \left(\frac{x}{g}\right)^2} \right] = \frac{V_{DD}^2}{2} \frac{C_0}{g} \left[ \frac{4x}{g(1 - \frac{x}{g})^2} - \frac{4x}{g(1 + \frac{x}{g})^2} \right] = \frac{2C_0}{g^2} V_{DD}^2 \alpha \]

In this way, we can notice the linear behavior of the electrostatic force with respect to the displacement performed by the seismic mass. Let’s consider a stationary acceleration applied to the sensor, in this case the general equation of the motion become

\[ m\ddot{x} + b\dot{x} + kx = ma_{ext} + F_{elec} \quad \rightarrow \quad m\ddot{x} + b\dot{x} + kx - F_{elec} = ma_{ext} \rightarrow x(k_{mec} + k_{elec}) = ma_{ext} \]

having defined

\[ k_{elec} = -2\frac{C_0}{g^2} \Delta V^2. \]

In our case \( k_{elec} = -1.4337 N/m \).

To obtain the value of the mechanical stiffness \( k_{mec} \), we can use the above motion equation, describing the external acceleration in terms of displacement and resonance frequency:

\[ x(k_{mec} + k_{elec}) = ma_{ext} \quad \rightarrow \quad x(k_{mec} + k_{elec}) = m\omega_0^2 \quad \rightarrow \quad \omega_0 = \sqrt{\frac{k_{mec} + k_{elec}}{m}} \]

With this data, the required mechanical stiffness becomes \( k_{mec} = 4.9617 N/m \).

Now we can start with the design of the springs. MEMS sensors usually features more than one spring. If the springs share the same ‘starting-point’ (different anchorages to the substrate are ideally the same point!) and the same ‘end-point’ (different ends connected to the same frame are ideally the same point!) then they are in parallel configuration, presented in Figure 1.
In this case, the total force applied to the system, \( F_{\text{tot}} \), can be expressed as the sum of the two elastic forces, \( F_1 \) and \( F_2 \). Moreover the displacement performed by the system is the displacement that each spring is subject to.

\[
F_{\text{tot}} = F_1 + F_2 = k_1 x + k_2 x = (k_1 + k_2) x = k_{\text{tot}} x
\]

Hence we can define the total stiffness \( k_{\text{tot,par}} \) of a system of \( n \) parallel springs as:

\[
k_{\text{tot,par}} = \sum_{m=1}^{n} k_m
\]

If the 'end-point' of a spring is the 'starting-point' of another one, they are in a series configuration, as shown in Figure 2.

In this case the total displacement of the system, \( x_{\text{tot}} \), is obtained by the sum of the displacements performed by both springs, \( x_1 \) and \( x_2 \). Besides, thanks to the action-reaction principle, the force applied to the system, \( F \), is applied to each spring. Then, we can obtain the formula to describe the total stiffness of a series configuration of springs, \( k_{\text{tot,ser}} \), as

\[
x_{\text{tot}} = x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2} = F \left( \frac{1}{k_1} + \frac{1}{k_2} \right) = \frac{F}{k_{\text{tot}}}
\]

From which we find:
\[ \frac{1}{k_{\text{tot,ser}}} = \sum_{m=1}^{n} \frac{1}{k_m} \]

As a tip for 'electronics guys', it can be said that equivalent stiffness is computed like the equivalent capacitance in an electrical network.

To define a 1-D displacement, 4 guided-end springs are designed in parallel and placed at the four corners of the frame, \( n_{\text{spring}} = 4 \). The elastic stiffness of each spring is:

\[ k_{\text{spring}} = Eh \left( \frac{w}{l} \right)^3 \]

In order to maximize the width of each spring the maximum available length for suspended springs is selected, see Table 1. In this way, we can obtain the maximum value \( w_{\text{max,GE}} \), since

\[ k_{\text{mec}} = n_{\text{spring}} k_{\text{spring}} = n_{\text{spring}} Eh \left( \frac{w_{\text{max,GE}}}{l_{\text{max}}} \right)^3 \rightarrow w_{\text{max,GE}} = l_{\text{max}} \sqrt[3]{\frac{k_{\text{mec}}}{n_{\text{spring}} Eh}} \]

obtaining \( w_{\text{max,GE}} = 1.4 \mu m \). This dimension is not allowed in this technology. In order to enlarge the springs, we can use a folded spring topology.

Figure 3: Four parallel guided-end springs.

Figure 4: Four parallel folded springs (two folds per spring).
To ‘fold a spring’ means to put \( n_{fold} \) springs of the same length in series. Given the required elastic stiffness, we can write the relation between the width of springs in the ‘folded’ topology and the width of ‘guided-end’ springs as a function of the number of folds, as

\[
k_{mec} = k_{folded} = k_{GE} \rightarrow \frac{n_{spring}}{n_{fold}} Eh \left( \frac{w_{fold}}{l_{max}} \right)^3 = n_{spring} Eh \left( \frac{w_{GE}}{l_{max}} \right)^3 \rightarrow w_{fold} = w_{GE} \sqrt[3]{\frac{n_{fold}}{n_{spring}}}
\]

knowing that \( w_{min} = 1.8 \mu m \), we can obtain the minimum number of folds as:

\[
w_{fold} = w_{GE} \sqrt[3]{n_{fold}} > w_{min} \rightarrow n_{fold} = \left( \frac{w_{min}}{w_{GE}} \right)^3 = 1.79 \rightarrow n_{fold} = 2
\]

then the corresponding value of the folded spring is \( w_{fold} = 1.75 \mu m \).

**QUESTION 4**

The motion of an accelerometer, and in general of each MEMS sensor, is affected by random ‘fluctuation force’, \( F_n \). In most cases this force arises from the interaction of the inertial mass with the residual gas particles in the MEMS cavity and the fluctuation force spectrum, \( S_{F,n} \), in unit of \([N^2/Hz]\) is:

\[
S_{F,n} = 4k_B T b
\]

The fluctuation force spectrum is independent of the frequency, than, it is a white noise. Now, we can obtain the expression for the noise equivalent acceleration density, \( NEAD \), through the transfer function that relates acceleration to force, \( T_{AF} \) (squared, because we are dealing with power spectral densities).

\[
(T_{AF})^2 = \frac{1}{m^2}
\]

\[
S_{A,n} = \frac{S_{F,n}}{m^2}
\]

\[
NEAD = \sqrt{S_{A,n}} = \sqrt{\frac{S_{F,n}}{m^2}} = \sqrt{\frac{4k_B T b}{m^2}} = \sqrt{\frac{4k_B T \omega_0}{mQ}} \rightarrow Q = \frac{4k_B T \omega_0}{mNEAD^2}
\]

In our case \( Q = 1.74 \).