In this class we will learn how to choose the focal length of an optic system, in order to satisfy the field-of-view requirement. We will then learn how to calculate the flux of photons that is captured by a single pixel of a digital camera embedded in a smartphone, and we will appreciate how this number depends on the distances between the light source, the scene and the lens, and on the optics; from the rate of incident photons, we will calculate the corresponding photocurrent. The photocurrent is be the signal contribution to the signal-to-noise ratio; its magnitude is thereby fundamental to evaluate the quality of an image. Finally, we will compare the intrinsic resolution of the sensor (due to spatial sampling) with the limit due to diffraction.

Problem

We are using our smartphone to take a picture of a basket of oranges. A lamp with an emitting optical power equal to 60 W is illuminating the scene, which reflects light as an ideal lambertian reflector; the reflectance coefficient is 0.2. The oranges are 2 meters away from the light source, and we are 10 meters away from the oranges. Compared to the direction orthogonal to the surface of the scene, we are tilted of 60°. From our point of view, the oranges occupy about one square meter. Our smartphone features an 8-megapixel sensor with 3260 x 2480 pixels. Each pixel has a size of 1.5 μm. The quantum efficiency of each pixel is equal to 0.6, while the transmittance of the red filter is equal to 0.85. The photograph is shot with an F-number equal to 4.

The situation is schematically sketched in Fig. 1.
We are asked to...

- Calculate the focal length that is best suited to fit the basket of oranges in the final picture.
- Calculate the rate of photons incident on one pixel.
- Calculate the photocurrent generated in a red pixel.
- Evaluate the resolution limit of the picture.

**PART 1: CHOICE OF THE FOCAL LENGTH**

We all know that the focal length of an optic system placed in front of an imaging sensor determines the magnification factor between an object on the scene plane and its image on the sensor plane. The focal length of an acquisition system is usually chosen to fulfill the field of view requirements.

The magnification factor of a lens can be evaluated as

\[ m = \frac{s_2}{s_1}, \]

where \( s_1 \) is the distance between the scene and the lens, and \( s_2 \) is the distance between the lens and the sensor. If \( s_2 \gg s_1 \), the magnification factor can be calculated as

\[ m = \frac{f}{s_1}, \]
where \( f \) is the focal length of the lens. The side of each pixel is equal to 1.5 \( \mu \text{m} \). From the number of pixels, we can infer the dimensions of the sensor. The height is

\[
H_{\text{sens}} = N_{\text{pix},H} \cdot l_{\text{pix}} = 2480 \cdot 1.5 \mu\text{m} = 3.72 \text{ mm},
\]

while the length is

\[
L_{\text{sens}} = N_{\text{pix},L} \cdot l_{\text{pix}} = 3260 \cdot 1.5 \mu\text{m} = 4.89 \text{ mm}.
\]

As we can see from Fig. 2, in order to fit the whole scene (one square meter) the sensor, one can chose the magnification factor as the one that guarantees that the highest dimension of the scene (1 m) corresponds to the lowest dimension of the sensor (3.72 mm):

\[
m = \frac{H_{\text{sens}}}{l_{\text{sens}}} = \frac{3.72 \text{ mm}}{1 \text{ m}} = 0.0037.
\]

Hence, the focal length of the lens should be

\[
f = s_1 m = 10 \text{ m} \cdot 0.0037 = 37 \text{ mm}.
\]

**PART 2: OPTICAL POWER**

The flux of photons impinging on a single pixel can be evaluated with the following procedure.

1. We will first calculate the area of the scene that corresponds to one pixel: only photons coming from this portion of the scene will be captured by the considered pixel.

2. Starting from the optical power of the source, we will evaluate the optical power that is impinging on the just-calculated surface.
3. We will evaluate the optical power that is reflected and its intensity.

4. We will finally calculate the optical power incident on the pixel. From this number, one can deduce the corresponding photon flux.

As the first step, we need to calculate the area of the scene that corresponds to the pixel, i.e. the area of the scene whose reflected photons, if captured by the lens, will be focused on the pixel. Photons reflected from other points of the scene will not reach the pixel. For a given pixel area $A_{\text{pixel}} = (1.5 \, \mu\text{m})^2$, if the scene was parallel to the lens plane, the corresponding area of the scene $A_{\text{SnPix},0}$ would be

$$A_{\text{SnPix},0} = \frac{A_{\text{pixel}}}{m^2} = \frac{(1.5 \, \mu\text{m})^2}{(0.0037)^2} = (0.4 \, \text{mm})^2.$$

In our situation, the camera system is tilted, with an angle $\beta = 60^\circ$ and the effective area of the scene (denoted as $A_{\text{SnPix}}$), whose reflected photons will be captured by the lens, is bigger. By simple geometry considerations, $A_{\text{SnPix}}$ can be calculated as

$$A_{\text{SnPix}} = A_{\text{SnPix},0} \cos(\beta) = 0.4 \, \text{mm} \cdot \frac{0.4 \, \text{mm}}{0.5} = 0.4 \, \text{mm} \cdot 0.8 \, \text{mm} = (0.57 \, \text{mm})^2.$$

Only photons coming from the area denoted with $A_{\text{SnPix}}$ will be focused of the pixel. Other photons will be focused in other pixels.

The next step consists in evaluating the optical power coming from the light source that is impinging on $A_{\text{SnPix}}$. This can be easily calculated with solid angles. Few words about solid angles. The solid angle defined by a point $P$ in the space and a portion of spherical surface $S$ belonging to the $P$-centered sphere with a radius $d$ is defined as (Fig. 3)

$$\Omega = \frac{S}{d^2}.$$
It is immediate to calculate the full solid angle \((4\pi \text{ sr, steradians})\), and the half solid angle \((2\pi)\). For small solid angles, the spherical surface portion \(S\) can be well approximated by the surface \(S_2\), whose area is easier to calculate since it’s a standard 2D area (Fig. 3):

\[
\Omega \approx \frac{S_2}{d^2}.
\]

Few words about the optical power of the source.
The source is said to emit 60 W of optical power. This value, in principle, is independent from the electrical power (i.e. 100 W) needed to keep the light source on. The electrical power is used to keep the light source working and to make it emit light. Of course, if the emission is efficient, most of the electrical power will be converted in optical power.
The optical power is proportional to brightness. The higher the optical power, the higher the brightness of the source. This is strictly true if the power spectral densities (PSDs) of the two sources are identical. Actually, the perceived brightness of an source/object is strongly wavelength-dependent as the responsivity of our eye depends on the wavelength (it has a peak around 550 nm). Hence, if the PSD of two light sources is different, it may happen that a low-power emitter is perceived as brighter than a high-power emitter, if the PSD of the former is centered in the green. For example, consider lasers: the closer their wavelength is to 550 nm, the brighter they will appear for a fixed optical power.

A light source is usually characterized with its power spectral density rather than with its optical power, as the reflectivity of objects or the transmittance of filters is usually wavelength-dependent, as well. The final optical power will depend on the integral of the overall power spectral density integrated on the whole visible range. Just for the sake of simplicity, we will not deal with PSDs, but we will directly use powers, to avoid lengthy integral computations.

As the light source is isotropic, its optical intensity (i.e. the optical power per unit solid angle) can be calculated as

\[
I_{\text{Src}} = \frac{P_{\text{Src}}}{4\pi} = \frac{60\text{ W}}{4\pi \text{ sr}} = 4.77 \text{ W/sr},
\]

since, for a point-like isotropic source, the emitted optical power is spread evenly over a sphere whose solid angle is \(4\pi\) steradians.

How much of this optical intensity drops on the portion \(A_{\text{SnPix}}\) of the scene? We should calculate the solid angle seen from the source towards \(A_{\text{SnPix}}\):

\[
\Omega_{\text{SnPix,Src}} = \frac{A_{\text{SnPix}}}{d^2} = \frac{(0.57 \text{ mm})^2}{(2\text{ m})^2} = 81.3 \text{ nsr}.
\]

The optical power impinging on \(A_{\text{SnPix}}\) is simply given by

\[
P_{\text{SnPix},I} = I_{\text{Src}} \Omega_{\text{SnPix,Src}} = 4.77 \text{ W/sr} \cdot 81.3 \text{ sr} = 388 \text{ nW}.
\]

Not all of the incident optical power is reflected: part of it might be absorbed, as in our case where the reflectivity is 0.2, i.e. only 20% of the incident photons are reflected:

\[
P_{\text{SnPix,R}} = P_{\text{SnPix,I}} R_{\text{Sn}} = 388 \text{ nW} \cdot 0.2 = 77.6 \text{ nW}.
\]
We should now evaluate where this luminous optical power is reflected and how much of this reflected power is captured by the lens. Let us first consider the simplest situation, where the reflection is isotropic. In this case, the reflected optical power per unit solid angle, i.e. the optical intensity, would be

\[
I_{R,\text{Iso}}(\beta) = \frac{P_R}{2\pi},
\]

uniform across all angles of the hemisphere of reflection.

In our case, the scene is said to have a lambertian-like reflectance, i.e. the reflected optical intensity is not uniform across all angles, but it follows a cosine law:

\[
I_{R,\text{Lamb}}(\beta) = I_{R,\text{Lamb},0} \cos(\beta).
\]

One can demonstrate that, for a given reflected optical power \(P_R\), the peak of the reflected intensity \(I_{R,\text{Lamb},0}\) is twice the intensity we would have with an isotropic emission:

\[
I_{R,\text{Lamb},0} = \frac{P_R}{\pi},
\]

and

\[
I_{R,\text{Lamb}}(\beta) = \frac{P_R}{\pi} \cos(\beta).
\]

This is due to the fact that the the 2D integral of the intensity over the hemisphere should give the total reflected power.

The two intensity profiles (isotropic and lambertian) are sketched in Fig. 4.

In our situation, we have that the optical intensity \(I_{\text{SnPix},R,\text{Lamb}}\) reflected by \(A_{\text{SnPix}}\) is

\[
I_{\text{SnPix},R,\text{Lamb}}(\beta) = \frac{P_{\text{SnPix},R}}{\pi} \cos(\beta).
\]

Since the lens is 60°-tilted with respect to the normal of the surface, the luminous intensity reflected by the portion \(A_{\text{SnPix}}\) directed in the direction of the lens is:

\[
I_{\text{SnPix},R,\text{Lamb},60^\circ} = \frac{P_{\text{SnPix},R}}{\pi} \cos 60^\circ = \frac{77.6 \text{nW}}{\pi \text{sr} \cdot 0.5} = 12.3 \text{nW/sr}.
\]
We can now evaluate the optical power impinging on one pixel. This value will be equal to the optical power that is captured by the lens, i.e. all photons coming from $A_{SnPi x}$ and captured by the lens will be focused on the considered pixel. This can be easily calculated by multiplying the reflected optical intensity for the solid angle seen from $A_{SnPi x}$ towards the lens. Note that this is an approximation: we are assuming that all the lens aperture is seen from the same angle $\beta$ from every captured point of the scene. This is an acceptable approximation, as (i) generally the distance between the scene and the lens is several times the lens aperture and (ii) the area $A_{SnPi x}$ of the scene, corresponding to the area captured by a single pixel on the sensor, is generally small compared to the distance $s_1$.

Since the focal length is 37 mm and the F-number is equal to 4, the lens diameter is

$$D_{Lens} = \frac{f}{F} = \frac{37 \text{ mm}}{4} = 9.3 \text{ mm},$$

the area of the lens is

$$A_{Lens} = \pi \left( \frac{D_{Lens}}{2} \right)^2 = (8.2 \text{ mm})^2,$$

and the solid angle seen from $A_{SnPi x}$ towards the lens is

$$\Omega_{Lens,SnPi x} = \frac{A_{Lens}}{s_1^2} = 680 \text{ nsr}.$$

The optical power impinging on the pixel is thus

$$P_{Pix} = I_{SnPi x,R,Lamb,\beta} \Omega_{Lens,SnPi x} = 12.3 \text{ nW/sr} \cdot 680 \text{ nsr} = 8.4 \text{ fW}.$$

**PART 3: CALCULATION OF THE PHOTON FLUX**

The rate of photons incident on the considered pixel can be calculated by dividing the incident optical power for the average photon energy $E_{ph}$ of the incident photons (650 nm corresponds to the orange color). We have

$$E_{ph} = \frac{hc}{\lambda} = \frac{6.626 \cdot 10^{-34} \text{ J} \cdot 3 \cdot 10^8 \text{ m/s}}{650 \text{ nm}} = 304 \cdot 10^{-21} \text{ J} = 1.9 \text{ eV}.$$ 

and thus

$$N_{ph} = \frac{P_{Pix}}{E_{ph}} = \frac{8.4 \text{ fJ/s}}{304 \cdot 10^{-21} \text{ J/photon}} = 27500 \text{ photons/s}.$$

We can calculate the photon flux per unit area, by simply dividing the previous number for the area of the pixel:

$$\phi_{ph} = \frac{N_{ph}}{A_{Pix}} = \frac{27500 \text{ photons/s}}{(1.5 \text{ \mu m})^2} = 12 \times 10^{15} \text{ photons/s/m}^2.$$
PART 4: CALCULATION OF THE PHOTOCURRENT

The quantum efficiency of the pixel $\eta_{pix} = 0.6$ is defined as the ratio between the number of photogenerated electron-hole pairs (EHPs) and the number of photons that reach the photodetector. In front of the considered pixel, an optical red filter is placed, whose transmittance coefficient is $T_{red} = 0.85$. Considering these losses, the photocurrent $i_{ph}$ can be calculated as

$$i_{ph} = q \eta_{pix} T_{red} N_{ph} = 1.6 \cdot 10^{-19} \text{C} \cdot 0.6 \cdot 0.85 \cdot 27500 \text{photons/s} = 2.25 \text{fA.}$$

The photocurrent could have been calculated through the responsivity $\mathcal{R}$ of the sensor, which relates the photocurrent and the optical power. Assuming an average wavelength of 650 nm,

$$\mathcal{R} = \eta \frac{q \lambda}{h c} = \frac{0.6 \cdot 1.6 \cdot 10^{-19} \text{C}}{6.626 \cdot 10^{-34} \text{J} \cdot 3 \cdot 10^8 \text{m/s}} \text{650 nm} = 0.31 \text{A/W},$$

and

$$i_{ph} = T_{filter} \mathcal{R} P_{pix} = 0.85 \cdot 0.31 \text{A/W} \cdot 8.4 \text{fW} = 2.25 \text{fA},$$

obviously the same as the one calculated above.

We can write the expression of the photocurrent by expanding all the used equations. We can start from the optical power incident on the pixel.

$$P_{pix} = I_{Sn pix, \Lambda, \beta} \Omega_{Lens, Sn pix} = \frac{P_{Sn pix, R}}{\pi} \cos(\beta) \frac{A_{Lens}}{s_1^2} = \frac{P_{Sn pix, R}}{\pi} \cos(\beta) \frac{\pi (D_{Lens})^2}{s_1^2}$$

$$P_{pix} = \frac{P_{Sn pix, 1 \pi R_{Sn}}}{\pi} \cos(\beta) \frac{\left(\frac{f_2}{2}\right)^2}{s_1^2} = I_{Src} \Omega_{Sn pix, Src \pi R_{Sn}} \cos(\beta) \frac{\pi (f_2/2)^2}{s_1^2}$$

$$P_{pix} = I_{Src} \frac{A_{Sn pix}}{d_1^2} \pi \cos(\beta) \left(\frac{f_2/2}{s_1^2}\right)^2$$

$$P_{pix} = \frac{A_{pix}}{m^2} R_{Sn} \pi \left(\frac{f_2}{2}\right)^2$$

$$P_{pix} = \frac{P_{Src} \cos(\beta)}{4 \pi d_1^2 R_{Sn}} \pi \left(\frac{f_2}{2}\right)^2$$

$$P_{pix} = \frac{P_{Src} A_{pix} R_{Sn}}{16 \pi d_1^2 F^2}$$
Hence, the photocurrent is

\[ i_{ph} = q \eta_{pix} T_{Red} N_{ph} = q \eta_{pix} T_{Red} \frac{P_{pix}}{E_{ph}}. \]

\[ i_{ph} = q \eta_{pix} T_{Red} \frac{\lambda}{hc} \frac{P_{Src} A_{pix} R_{SN}}{16\pi d_{1}^{2} F^{2}}. \]

Few considerations are reported.

- An increase in the quantum efficiency and in the transmission coefficient of the color filter would be beneficial for the signal.

- The photocurrent generated in one pixel is proportional to the inverse of the square of the F-number. A change in the F-number from 4 to 11 would have decreased the photocurrent by a factor \( \approx 8 \).

- A bigger pixel would lead to a higher photocurrent.

- The optical power is independent from the angle \( \beta \). This is in agreement with the fact that the scene is a lambertian reflector: the incident photon flux is the same, independently on the angle; in fact, when increasing the angle, the intensity decreases, but the projected area increases by the same amount.

- Do not be fooled by the dependence on the \( \lambda \) coefficient: a higher wavelength does not always lead to a higher current! We all know that given a flux of photons, the photocurrent is independent from the wavelength: one photon means one EHP. The dependence from the wavelength is real when considering a fixed optical power: the same optical power corresponds to different photon fluxes if the wavelength is different.

**PART 5: RESOLUTION**

We can calculate the dimension of the first Airy circle, that determines the diffraction-limited resolution:

\[ d_{Airy} = 2.44 \cdot \frac{\lambda}{D_{f}} f = 2.44 \cdot \lambda F = 2.44 \cdot 650 \text{ nm} \cdot 4 = 6.3 \text{ \( \mu \)m}. \]

The diffraction-limited resolution turns out to be lower\(^1\) than the pixel size (1.5 \( \mu \)m). The resolution of this sensor is thus diffraction limited! This means that, for this sensor size (and this F-number), increasing the number of pixels does not improve the resolution!

\(^{1}\)Be careful using *high* and *low* when addressing to the resolution, use *better* or *worse* to avoid misunderstandings. In this case the Airy disk turns out to be higher than the pixel size, thus the resolution is lower/worse.